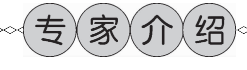
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近年来研究方向包括人工边界、自动化网格剖分、基于图像的分析、土－结构动力相互作用、结构动力学与地震工程、波动传播、断裂力学、弹塑性损伤模型和高性能计算等方面。宋崇民教授2009 年以来主持澳大利亚科学理事会(Australian Research Council)探索项目(Discovery Project)8 项,联合项目(Linkage Project)5 项。于 Computer Methods in Applied Mechanics and Engineering、International Journal for Numerical Methods in Engineering、Engineering Fracture Mechanics、Computers & Structures、Engineering Analysis with Boundary Elements、International Journal of Mechanical Sciences 等国际顶级学术期刊发表论文 200 余篇,会议论文 80 余篇,多次在重大国际会议做特邀报告,出版专著 2 部,培养博士生 20 余人。

比例边界有限元法的最新研究进展

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摘　要:随着计算机技术的进步,数值计算方法在工程领域得到了广泛的应用。有限单元法(FEM)作为其中的典型代表,已发展为成熟的商业软件。然而,现有的有限元方法在无限域动力响应问题、应力集中问题和网格自动化剖分等方面仍存在一定的局限性。此外,日新月异的建模技术引入了大量新型几何模型格式,例如电子扫描图像、3D 打印模型、点云模型等,都对数值模拟带来了新的挑战。比例边界有限元法(SBFEM)作为一种新型的半解析数值方法,融合了有限元方法(FEM)和边界元方法(BEM)的优势,只在单元表面进行离散化,而在单元内部进行解析求解,可将问题的维度降低一维,在处理无限域动力问题和带有奇异性的应力集中问题时具有独特的优势。SBFEM

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单元仅需满足可见性要求,提高了单元形状的灵活性,可以构造包含任意数量节点的多边形和多面体单元,结合八叉树等高效自动化网格剖分算法,可以实现与多种几何模型的无缝对接,适合大规模并行计算。近年来,SBFEM 已经发展成为一种可满足现代工程计算需求的通用高效计算工具,在无限域动力问题、断裂力学、非线性问题、接触、自适应分析、反演问题、多场耦合、高性能计算等方面显示出了巨大的应用前景。本研究重点对 SBFEM 的发展历程和近期研究热点进行系统地综述,并展望未来发展趋势,为相关领域的研究人员和工程技术人员提供参考。

关键词:比例边界有限元法;无限域;应力强度因子;多面体单元;八叉树网格;高性能计算

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The latest progress in research on the scaled boundary finite element method

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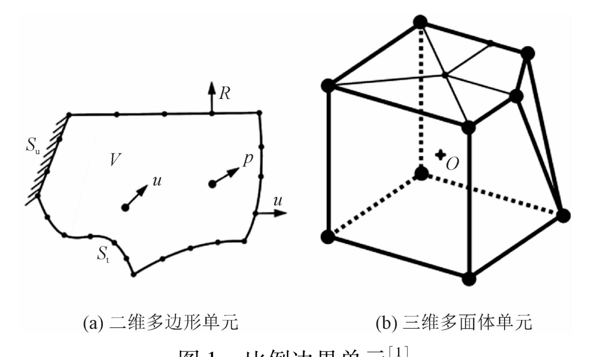
Abstract: With the rapid development of modern computers, the numerical method has been widely used in engineering. The finite element method (FEM), one of the most popular methods, has been successfully integrated into many commercial software packages. However, there are still some limitations in the current FEM, including the modeling of dynamic response of unbounded domain, the representation of stress singularities, and the automatic mesh generation. Additional challenges are posed by the emerging data formats used in modern geometric modeling techniques, such as digital image, stereolithography (STL) models, and point cloud models. The scaled boundary finite element method (SBFEM) is a novel semi-analytical method combining some of the advantages of FEM and boundary element method (BEM). It only discretizes the boundary of the element, while the radial direction is solved analytically, reducing the spatial dimension by one. As a result, the SBFEM is perfectly suitable for the modelling of dynamic stiffness of unbounded domain and representing stress singularities. Furthermore, it is capable of modeling polyhedrons with an arbitrary number of faces and nodes, which greatly reduces the difficulty associated with mesh generation when combining with efficient octree algorithm. Over the years, it has been developed as a general-purpose numerical method, and a large number of researchers have endeavored to apply the SBFEM in fields such as dynamic response of unbounded domain, fracture mechanics, nonlinear analysis, contact problem, adaptive analysis, inverse problem, and high performance computing. In this article, the history and latest progress in SBFEM research are systematically reviewed.

Key words: scaled boundary finite element method; unbounded domain; stress intensity factor; polyhedral element; octree mesh; high performance computing

随着计算机技术的飞速发展,数值计算方法在航天、机械、材料、土木等多个工程领域得到了广泛的应用。有限单元法(finite element method, FEM)作为其中的典型代表,已发展为成熟的商业软件。然而,现有的有限元方法在一些问题中仍存在局限性,例如无限域的动力响应、应力场的奇异性和自动化网格剖分问题等。在计算无限域的动力响应时,需要满足辐射边界条件,即能量不能从无穷远处传播到结构。因此通常需要构建巨大的计算域,计算量随之大幅增加。在断裂力学问题中,裂纹尖端存在奇异的应力场,FEM 由多项式表示的形函数难以准确描述,一般需要快速单元尺寸过渡。FEM 需要将计算机辅助设计(CAD)模型离散化为形状简单的单元,即网格剖分过程,该过程往往需要大量人力操

作,生成的网格质量高度依赖分析人员的技能和经验,因此耗时且易出错。此外,新兴的数字化建模技术引入了大量与传统 CAD 不同的几何模型格式,例如电子扫描图像、3D 打印模型、点云模型等,都给数值模拟带来了新的挑战。

比例边界有限元法(scaled boundary finite element method,SBFEM)作为一种新型的半解析数值方法,旨在克服传统 FEM 方法的一些局限性。SBFEM 起源于对无限域动力刚度的研究。1997 年,Song 和 Wolf 在求解无限域的动力响应问题时,首先提出了 SBFEM 的概念[1]。该方法借鉴了边界元法的思路,只在计算域的边界进行离散化,将问题的维度降低一维,辐射边界条件自然满足,而不需要基本解(图 1)。在单元内部,计算变量的分布可以由固体力学控制方程直接得出,不需要引入其他假设。随后,Wolf 和 Song 又提出了 SBFEM 单元的动力刚度矩阵在频域下的解析解[2]和基于单元脉冲响应的时域化方法[3]。1998 年,在布宜诺斯艾利斯举行的第四届世界计算力学大会上,Wolf 和 Song 完整地展示了 SBFEM 方法[4],获得了广泛关注,从此 SBFEM 的理论发展进入了黄金时期。Song 在 1999 年研究了体力存在时非齐次方程的求解,从而将 SBFEM 的适用范围扩展到不同分布的体力情况[5]。在 2000 年,两篇关键性的文章由 Song 和 Wolf 发表,分别详细阐述了比例边界单元的推导[6]和求解过程[7]。其后,Deeks 和 Wolf[8]给出了 SBFEM 在弹性动力学中基于虚功原理的推导,Song[9]研究了 SBFEM 方程在静力条件下的矩阵形式解。至此,SBFEM 的基本理论发展已经逐渐成熟,在无限域动力问题中体现出了巨大的应用价值。



(a) 二维多边形单元　　(b) 三维多面体单元

图 1　比例边界单元[1]

Fig. 1　Scaled boundary finite element[1]

其后,研究者发现 SBFEM 在复杂材料界面和包含奇异性问题中具有独特的优势[10-14]。由于 SBFEM 的开放式多边形单元的形函数包含奇异项,可以准确表示不同材料界面处裂纹尖端的应力集中系数[15]和 T-应力[16],且该系数可通过 SBFEM 单元求解过程直接获得[17],极大地简化了奇异性问题的求解过程,提高了计算精度。

随着 SBFEM 的快速发展,在单元的边界上,各种新型表面单元也被相继引入用于离散边界,包括高阶谱单元[18]、样条曲线[19-21]等,在板单元的研究方面也取得了显著进展[22-26]。

伴随多边形[27-28]和多面体单元[29-31]的开发和完善,SBFEM 逐渐被广泛认为是一种更通用的数值方法[28,32],可以解决数值计算中的各类问题。由于多边形和多面体单元形状的灵活性,SBFEM 可以采用四叉树和八叉树等高效的网格剖分技术[33-35],极大提高网格剖分的效率和自动化程度。

近年来,SBFEM 与八叉树网格结合,在高性能计算领域展现出了巨大的应用前景,充分利用并行计算机的多核运算性能,大幅度提高了计算效率[36-37]。经过多年的发展,SBFEM 的理论日臻完善,算法逐渐成熟,在各个工程领域的应用引人瞩目,并初步实现了与商用软件的集成[38-42]。

本研究旨在总结 SBFEM 的发展历程和近期研究热点,为 SBFEM 领域的研究人员和工程技术人员提供可供参考的学习思路和全面的资料索引,同时也方便各领域相关研究的人员拓展思路和视野,了解最新的研究和应用进展。

1　基本理论及特征

由于篇幅所限,本节只简要给出 SBFEM 的一些基本公式和特征。完整 SBFEM 坐标变换、单元公式推导和求解过程可参考 Song 关于 SBFEM 的专著[43]。

1.1　基本理论

SBFEM 是一种半解析方法,仅需要离散单元表面,在径向进行解析求解。图 1(b)展示了一个典型的比例边界多面体单元,该单元的边界可包含任意数量的面、边和节点。单元只需要满足可见性要求,即可以在单元内定义比例中心 O,整个单元的边界对 O 直接可见。将比例中心到边界的连线定义为比例中心线,并用比例边界坐标 ξ 表示,该坐标在有限域取[0,1],在无限域为[1,+∞)。在边界上通过二维等参单元(η,ζ)对边界进行离散化。该单元的节

点坐标可以表示为 $\boldsymbol{x}$。整个多面体单元可通过将边界连续向比例中心缩放的方式表示。多面体内的任意一点坐标 $\hat{\boldsymbol{x}}$ 可以通过比例边界坐标 ξ 和边界单元的形函数 $\boldsymbol{N}(\eta,\zeta)$ 表示为

$$\hat{\boldsymbol{x}}(\xi,\eta,\zeta)=\xi\boldsymbol{x}_b=\xi\boldsymbol{N}(\eta,\zeta)\boldsymbol{x} \tag{1}$$

其中 $\boldsymbol{x}_b$ 表示在同一条比例中心线上的边界单元上的点的坐标。在 O 与节点连接的比例中心线上引入未知位移函数 $\boldsymbol{u}(\xi)$，见图 2，单元内任意点的位移可以在同一 ξ 坐标处通过形函数插值得出，即

$$\boldsymbol{u}(\xi,\eta,\zeta)=\boldsymbol{N}^u(\eta,\zeta)\boldsymbol{u}(\xi) \tag{2}$$

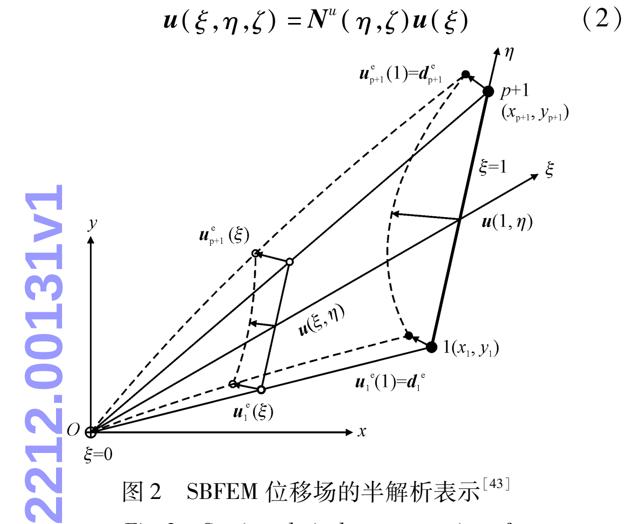


图 2　SBFEM 位移场的半解析表示[43]

Fig. 2　Semi-analytical representation of displacement field in SBFEM[43]

将上述坐标变换代入虚功原理，通过一些数学处理，可将固体力学的控制偏微分方程转换为一组关于比例边界坐标 ξ 的二阶常微分方程，即比例边界有限元方程

$$\boldsymbol{E}_0\xi^2\boldsymbol{u}(\xi)_{,\xi\xi}+(\boldsymbol{E}_0+\boldsymbol{E}_1^{\mathrm{T}}-\boldsymbol{E}_1)\xi\boldsymbol{u}(\xi)_{,\xi}-\boldsymbol{E}_2\boldsymbol{u}(\xi)=\boldsymbol{0} \tag{3}$$

其中，系数矩阵 $\boldsymbol{E}_0$，$\boldsymbol{E}_1$ 和 $\boldsymbol{E}_2$ 只与离散后边界的几何形状与材料属性有关，可通过表面数值积分的方式求解[43]。该方程可进一步转换为一阶常微分方程，并通过特征值分解的方式求解，分解后获得的特征向量矩阵可以直接计算单元刚度矩阵 $\boldsymbol{K}$，即

$$\boldsymbol{K}=\boldsymbol{\varphi}_q\boldsymbol{\varphi}_u^{-1} \tag{4}$$

其中 $\boldsymbol{\varphi}_q$ 和 $\boldsymbol{\varphi}_u$ 均为特征向量矩阵的子阵。单元质量矩阵 $\boldsymbol{M}$ 的求解方法类似。整体刚度和质量阵的组装过程与 FEM 相同。

对于非线性问题，可利用 SBFEM 方程构造多边形和多面体单元的形函数 $\boldsymbol{N}_V$，即

$$\boldsymbol{N}_V=\boldsymbol{N}^u(\eta,\zeta)\boldsymbol{\varphi}_u\xi^{\boldsymbol{\lambda}}\boldsymbol{\varphi}_u^{-1} \tag{5}$$

其中 $\boldsymbol{\lambda}$ 为包含特征值的对角矩阵。通过此形函数可用节点位移表示单元内任意一点位移（图 3），可以同 FEM 一样进行非线性计算[44-45]。

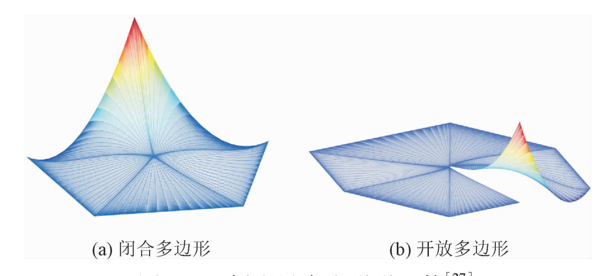


图 3　比例边界多边形形函数[27]

Fig. 3　Scaled boundary polygonal shape function[27]

1.2　主要特征

①满足单位分解和线性完备。SBFEM 形函数可以精确表示刚体运动和常应变状态，增加泡沫函数后也可满足任意阶完备[28]。

②在边界上可以应用任意高阶谱单元，也可以混合不同类型单元，仅需满足在边界上的连续性[18]。

③在边界上采用标准数值积分。可以采用与 FEM 相同的高斯积分或 Gauss-Lobato-Legendre 积分策略。

④开放式单元的形函数包含奇异性，可以对不同阶的奇异性精确求解[15-16]。

⑤施加边界条件的方式与 FEM 相同，可以被视为一种通用的数值方法。

⑥降低网格剖分难度。由于仅离散边界，而在单元内部解析求解，因此单元形状具有很高的灵活性，仅需要满足可见性要求[46]，适合与四叉树（2D）、八叉树（3D）网格剖分算法结合，效率高，鲁棒性好。

⑦适用于高性能计算。SBFEM 与基于模板单元的八叉树网格结合，可以大量减少对计算机内存的需求，适合开发并行算法，结合预计算单元矩阵和对角化质量阵，大幅提升超大型问题的计算效率[36-37]。

2　研究进展

本节简要列出 SBFEM 在若干领域的发展历程。

2.1　无限域动力刚度的计算

2.1.1　理论研究

SBFEM 最早的应用场景是无限域动力刚度的计算[1]。SBFEM 单元通过将边界向外扩张，可以准确、简洁地构建出无限域。早期研究集中在频域下

的动力刚度计算[47-50]。借助于 Birk 和 Song 提出的基于连分式展开的高精度方法[51-52]，Bazyar 和 Song 提出了在二维问题上的高精度时域化的 SBFEM 方法[53]。Chen 等[54]运用 SBFEM 研究了各向同性土体中的波传导，其中单元脉冲响应方法被用于动力刚度的时域化过程。在此基础上，Chen 等通过分块矩阵等技术进一步降低了时域化计算成本[55]，并成功扩展到波在复杂土层结构中的传播[56]。Gravenkamp 等运用 SBFEM 计算结构中的波动传播，用于检测长结构中的缺陷[57]。林皋和刘俊运用 SBFEM 研究了波导本征问题分析[58-59]。

2.1.2　工程应用

SBFEM 在无限域的理论研究促进了一大批工程应用，包括土-结构相互作用[54]、大坝-水库相互作用[47]等，在抗震计算领域取得了丰硕的成果。Gong 等[60]研究了无网格法-SBFEM-FEM 耦合的土体-结构相互作用算法。Wang 等[61]提出了重力坝和水域相互作用的时域化计算方法，其中双向渐进技术被用于精确模拟高度恒定的无限水域。Qu 等[47]提出了基于二维 SBFEM 水平半空间无限水域、半空间无限土体和重力坝结合（图 4b）的数值计算方法。

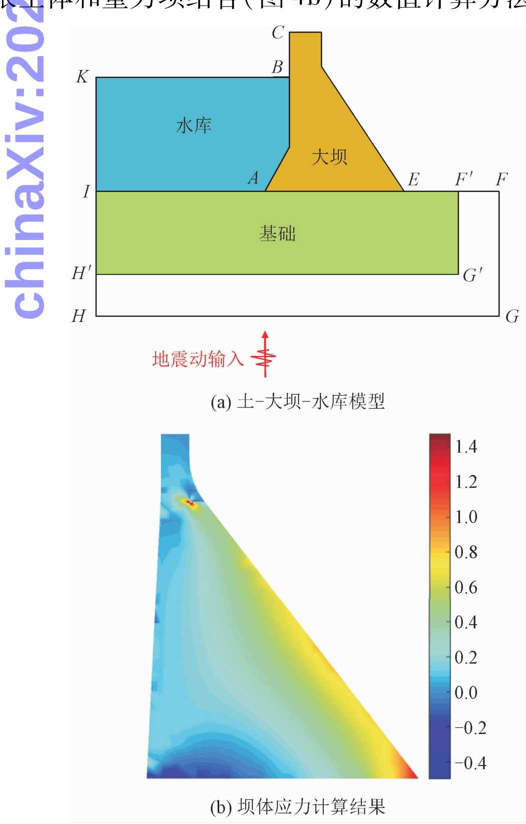


图 4　SBFEM 无限域单元在工程中的应用[47]

Fig. 4　The application of SBFEM unbounded domain[47]

许贺等[62]研究了基于 FEM-SBFEM 的大坝-水库动力耦合简化分析方法。Zhong 等[63]提出了大坝-基础界面在水压力作用下的断裂模型。Qu 等[64]研究了三维 SBFEM 空间模型外源激励时的地震波传播规律。Li 等[65]将 SBFEM 应用于成层半空间的波动传播过程分析，并成功应用于计算拱坝基础的动力刚度[66-67]。Lu 等[68]研究了倾斜土层的时域化问题。Xu 等[69]研究了大坝和可压缩水体的非线性动力相互作用。

2.2　断裂和损伤问题

断裂和损伤问题一直以来都是计算固体力学中的重要难点问题。由于裂纹尖端存在奇异性，基于多项式的有限元形函数难以准确描述其应力分布。在裂纹扩展过程中，由于几何的不断变化，网格也需要不断更新，带来了更大的挑战。SBFEM 单元的独特优势提供了研究断裂问题的新思路，该方面的综述可参见 Song 的综述[14]。

2.2.1　应力集中系数

基于 SBFEM 的开放式多边形单元的形函数包含奇异项（图 5a），可以准确计算应力集中系数，且该系数可通过 SBFEM 单元求解过程直接获得[70]。

Song 等[15]提出了基于半解析奇异应力场的应力集中系数，可应用在多种材料界面，并适用于任意种类的奇异性。Chidgzey 和 Deeks 研究了常用的断裂参数，包括应力集中系数、T-应力和高阶系数等，均可以通过 SBFEM 方程直接获取[17]。Li 等[71-72]运用该方法研究了压电材料的应力集中。Sun 等[73]研究了凹痕和孔洞附近的应力集中系数，非均匀的多边形单元极大的提高了网格剖分的灵活性。Egger 等[74]提出的舒尔解耦和超级收敛恢复技术进一步提高了计算应力集中系数的自动化程度和精度。庞林等[75]提出了比例边界等几何法，准确计算了应力集中系数。SBFEM 已被广泛应用于估算多层材料孔洞附近的应力[76]，静力和动力作用下复合材料多层板的应力集中系数[77]，以及预测两种材料界面处的裂纹扩展方向[78]。在三维问题中（图 5b），SBFEM 的优势依然十分明显，近期的研究成果包括各向同性介质界面[79-81]，复合多层材料[82-84]，和压电材料[85]中的断裂问题。

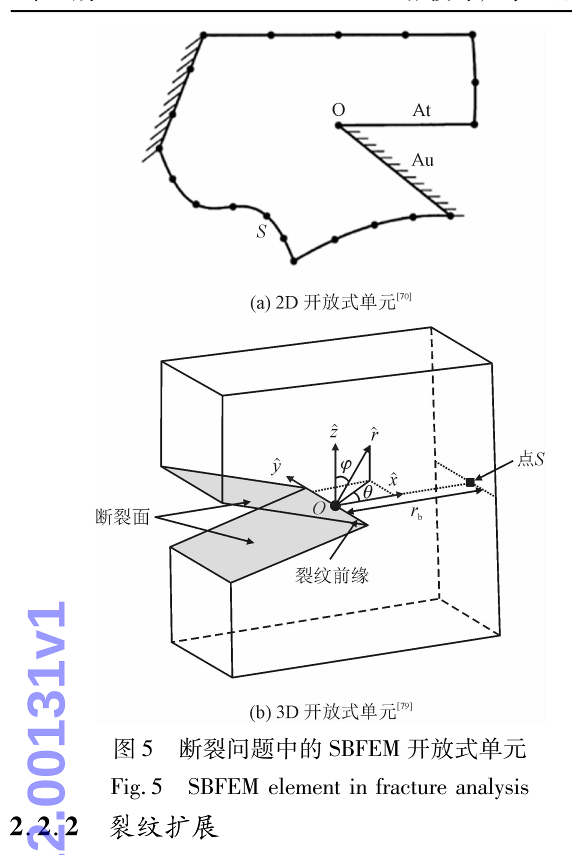


图 5　断裂问题中的 SBFEM 开放式单元

Fig. 5　SBFEM element in fracture analysis

2.2.2　裂纹扩展

得益于四叉树网格可局部修改的特性，SBFEM 可以便捷地模拟裂纹扩展过程，而不需要在每一步重新划分全部网格。

一种基于 SBFEM 的裂纹扩展模型由 Yang 等提出[86]。其原理随后被应用于混凝土的非线性黏结断裂[87-91]和动力学断裂[92-93]。Ooi 等[94]提出了基于多边形单元的全自动裂纹扩展模拟算法（图 6），还提出了混合多边形网格和四叉树网格的新型方法[33,95-96]。Jiang 等[97]运用 SBFEM 和四叉树网格研究了细观尺度下混凝土的开裂模型。Iqbal 等[98]研究了热-力耦合作用下的裂纹扩展。Guo 等[99]研究了基于四叉树和多边形网格的混凝土细观断裂模型。Qu 等[100]研究了地震作用下大坝的断裂。刘钧玉等[101]研究了重力坝-地基-库水系统的动态断裂。

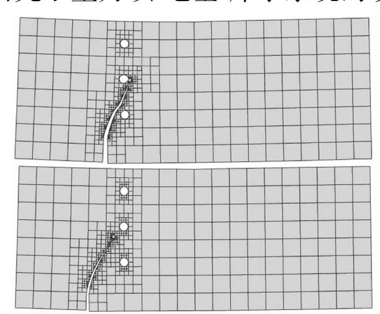


图 6　SBFEM 多边形单元模拟裂纹扩展[94]

Fig. 6　Crack propagation using polyhedral element in SBFEM[94]

李建波等[102]应用基于水平集算法的扩展比例边界有限元法研究了裂纹扩展问题。

2.2.3　损伤

Zhang 等[103-104]研究了 SBFEM 非局部损伤模型中的应用（图 7），消除了网格敏感性，并在损伤区域用四叉树网格进行了局部网格细分。该方法已被推广到三维问题[105]，并被用于研究细观尺度下混凝土的损伤[106]。杜成斌等[107]研究了 SBFEM 在非局部宏观与微观尺度下的损伤模型并进行开裂模拟。Ankit 等[37]将非局部损伤模型并行化，极大地提高了计算效率。

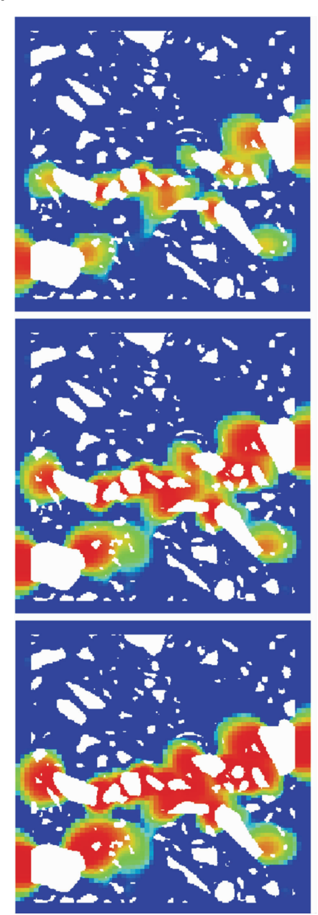


图 7　SBFEM 研究复合材料损伤问题[103]

Fig. 7　Damage analysis of composite material using SBFEM[103]

Hirshikesh 等[108]研究了相场模型在断裂力学中的应用。Natarajan 等[109]将相场模型与 SBFEM 结合研究了脆性材料在动力作用下的断裂。Assaf 等[110]将该方法推广到三维问题，并应用八叉树网格实现自适应细分。

2.3　板壳单元

板壳单元相对于实体单元额外包含了转动自由

度，可以准确描述结构构件的弯曲变形特性，在结构力学计算中有广泛的应用。Gravenkamp 等[111]研究了 SBFEM 构建的板单元，并在多层材料的波动传播中取得了进展，高阶单元也在其中得到了应用。Man 等[112]统一了板单元与三维实体单元的构建形式(图 8a)，相比于传统结构单元，SBFEM 板单元提高了计算结果精度，避免了剪切自锁现象[113-114]。Xiang 等[115]研究了板的屈曲分析。随着各种新材料的涌现，SBFEM 在多层复合三明治板[22]、功能梯度板[25,116-117]、压电材料板[26]、多孔板[118]等的弯曲和屈曲分析以及磁场-电场耦合作用[24]中都取得了重要进展。Liu 等[119]研究了圆柱形板的弯曲和震动特性。Wallner 等[120]将该方法推广到 SBFEM 壳单元(图 8b)。Li 等[121-122]运用 SBFEM 高阶壳单元研究了功能梯度材料的动力和静力问题。

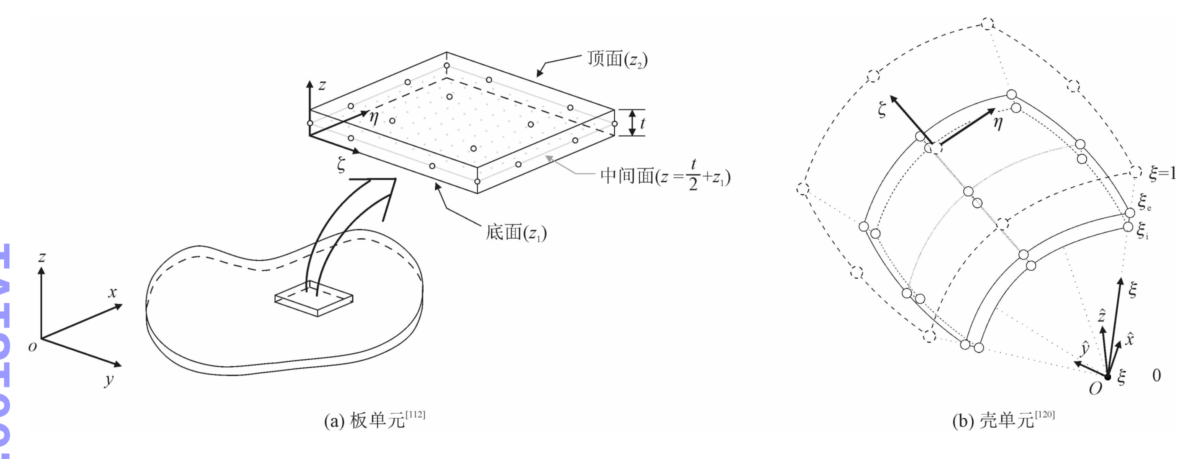


图 8 SBFEM 结构单元

Fig. 8 SBFEM structural elements

2.4 材料非线性计算

Ooi 等[44]提出了基于多边形 SBFEM 单元的弹塑性计算方法，在比例边界线上引入附加积分点，实现了对材料非线性属性的高精度拟合。He 等[123]进一步简化了弹塑性算法，只取比例中心一个点为积分点进行非线性拟合，提高了计算效率。Liu 等[45]将弹塑性计算方法扩展到三维问题并成功与八叉树网格结合(图 9)。Eisenträger 等[124]针对混凝土材料徐变开发了基于电子扫描图像的非线性分析方法，并考虑了温度的影响。Jabbari 等[125]运用 SBFEM 分析非均质和各向异性的非线性土体本构模型，用于解决地质问题。Chen 等[126]将 SBFEM 和 FEM 结合，研究了地铁车站的弹塑性损伤破坏机理。

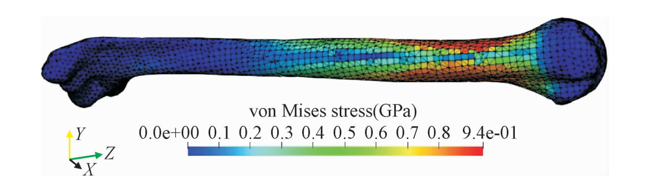
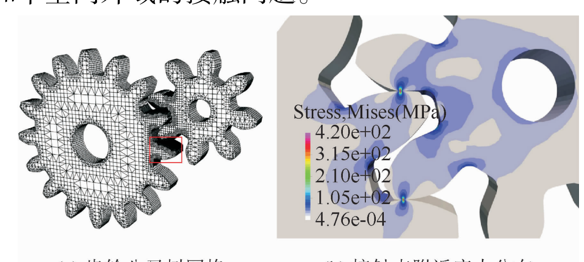


图 9 SBFEM 三维弹塑性计算[45]

Fig. 9 3D elastoplastic analysis using SBFEM[45]

2.5 接触问题

Zhang 和 Song[127]提出了将非匹配网格转化为匹配网格并在界面处做局部优化处理的技术，提高了多个子结构网格独立生成的灵活性。Xing 等[128-129]将该方法应用于二维和三维(图 10)的接触问题，将两侧网格匹配后，可用点对点的接触理论计算，降低了问题的求解难度。Zhang 等[130]将 SBFEM 应用于开裂面的接触分析，并研究了静力和动力作用下的裂纹扩展问题[131]。Chen 等[132]研究了结构和半空间外域的接触问题。



(a) 齿轮八叉树网格 (b) 接触点附近应力分布

图 10 SBFEM 非匹配网格在接触问题中的应用[129]

Fig. 10 Contact problem in SBFEM using non-matching meshes[129]

2.6 自适应分析

自适应是在分析中根据误差估计局部细分网格

的迭代求解技术,SBFEM 与八叉树网格的结合为自适应分析提供了独特的优势。SBFEM 在二维自适应分析中的应用可追溯到 Deeks 的研究[133-134],该方法仅需局部细分单元的表面,而单元内部保持不变,降低了计算成本。近年来,Song 等研究了二维四叉树网格的自适应划分[135],Zhang 等基于多树细分技术研究了三维多面体网格的自适应细分[136](图 11),相较于均匀网格细分,自适应算法提高了计算结果精度,降低了计算成本。Nie 等[137]利用多树细分技术研究了边坡稳定性问题。

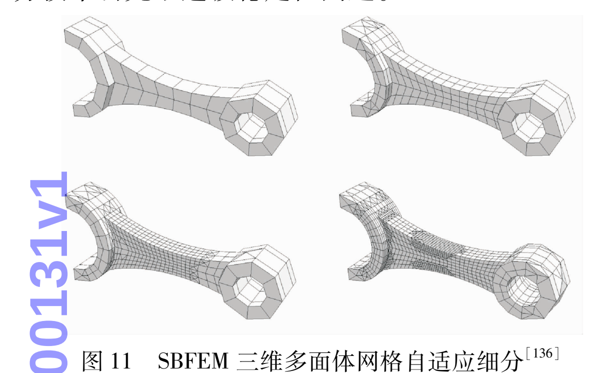


图 11　SBFEM 三维多面体网格自适应细分[136]

Fig. 11　Adaptive refinement using SBFEM polyhedral elements[136]

2.7　反演问题和结构拓扑优化

反演问题利用已知荷载和观测到的响应反推结构,在结构健康检测等领域有广泛的应用。余波等利用 SBFEM 研究了检测孔洞和裂纹的技术[138-139]。江守燕等[140]将 SBFEM 与深度学习算法结合,研究了结构内裂纹状缺陷反演的技术,并推广到薄板结构[141-142]。Li 等[143]基于 SBFEM 研究了不可压缩材料的拓朴优化技术。Zhang 等[144]研究了三维结构拓扑优化,自适应技术被用于降低优化计算成本。

2.8　其他物理问题

除了传统力学问题,SBFEM 也在热学、声学、电磁学等多个物理问题领域得到了广泛应用。

2.8.1　渗流

Liu 等[20]将边界等几何表示与 SBFEM 结合,研究了包含复杂几何的渗流问题。Zou 等[145]将固体力学控制方程与渗流理论结合,使网格节点既包含位移自由度,又包含孔隙压力自由度,研究了 SBFEM 在三维含水多孔介质中的应用。

(a) 热传导问题[146]　(b) 声固耦合分析[147]

(c) 饱和多孔介质问题[145]　(d) 静电场分布[148]

图 12　SBFEM 在其他物理问题中的应用

Fig. 12　SBFEM in analysis of other physical problems

2.8.2 热传导

He 等[149-150]研究了基于图像的二维 SBFEM 热传导,结合自适应技术,可预测复合材料的等效导热系数。Yu 等[151]将 SBFEM 应用于热传导瞬态分析,结合四叉树网格提供了便捷高效的计算方法。Li 等[146]用基于样条曲线的 SBFEM 研究热传导问题。

2.8.3 声学

Birk 等[152]研究了 SBFEM 在二维无限域中的声学问题。Liu 等[153]将该方法推广到三维,利用 SBFEM 构造的无限域研究声场特性,并进一步基于非匹配网格开发了声-固耦合算法[147]。

2.8.4 电学

Liu 等[148]将 SBFEM 用于研究静电场的分布。Liu 等[154]用 SBFEM 研究了电磁场问题。Li 等研究了压电材料在力学[72]和温度荷载[71]作用下的断裂问题。Sladek 等[155]研究了压电材料的断裂特性,Saputra[156]在此基础上结合四叉树网格生成技术,将其应用于压电材料的电子扫描图像,进行了力-电耦合分析。张勇等[157]提出了移动相似中心的比例边界有限元方法解决静电场问题。

2.9 新型几何模型

随着建模技术的发展,越来越多传统 CAD 模型之外的新型几何模型格式给数值分析带来了额外的挑战,其中最具有代表性的是电子扫描图像、3D 打印模型(stereolithography,STL)和点云模型。

2.9.1 电子扫描图像

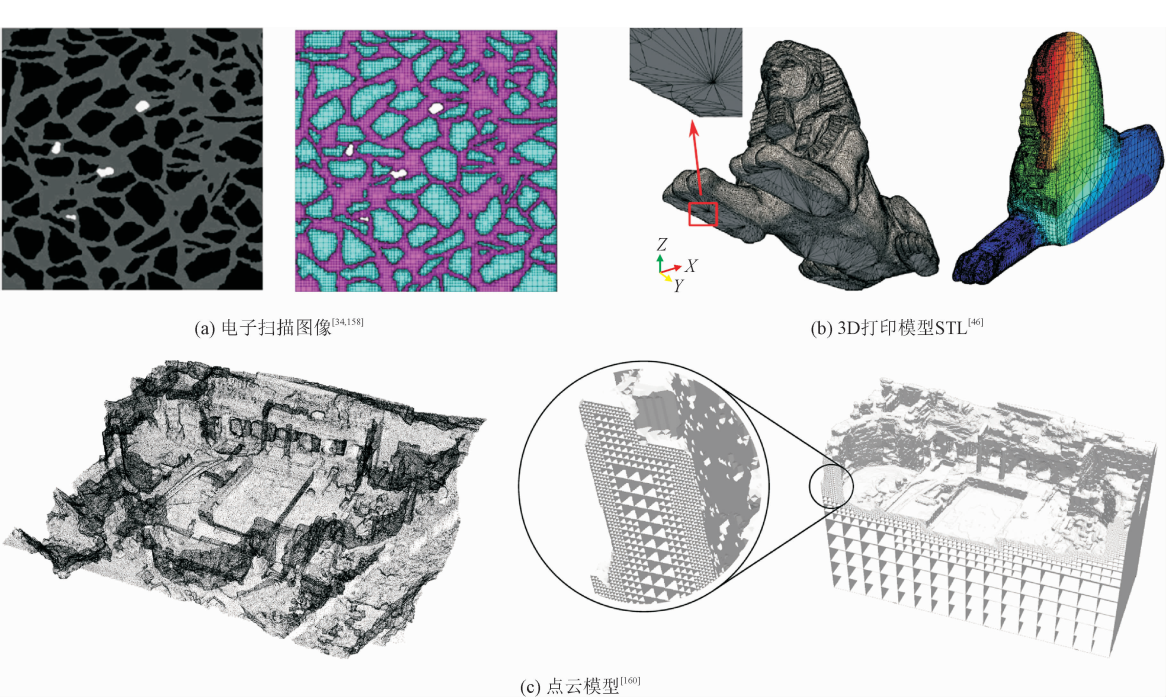
Saputra 等[34]等开发了基于电子扫描图像的网格剖分算法,可以快速生成平衡八叉树网格(图 13a),成功应用于混凝土材料的细观力学性能分析(模型来源于 Huang 等[158])。其后又进一步研究了复合材料的等效材料属性[29]。Gravenkamp 等[159]将 SBFEM 和 transfinite 单元结合,用于基于电子扫描图像的数值分析。

2.9.2 3D 打印模型 (STL)

Liu 等[46]提出了一种基于 STL 格式的八叉树网格剖分技术,生成背景网格后在边界按照 STL 模型进行切割,生成与几何模型边界一致的多面体网格(图 13b)。该方法可以在包含应力集中的边和角处构建开放式单元,可以准确描述应力集中。该方法精确、高效、鲁棒性好,被广泛应用于近年来的 SBFEM 研究成果。

2.9.3 点云模型

Zhang 等[160]研究了基于点云模型的八叉树网格自动化生成方法。该方法只需要离散化的带方向点云模型(图 13c),不需要表面重构,因此极大地降低了人工操作的难度。该方法生成的网格基于八叉树模板单元,适合进行并行计算。



(a) 电子扫描图像[34,158]　(b) 3D打印模型STL[46]　(c) 点云模型[160]

图 13 SBFEM 基于新型几何模型的数值计算

Fig. 13 SBFEM analysis based on new geometric model

2.10　新材料

SBFEM 独特的单元灵活性使其在新材料的力学性能计算方面获得了极大地关注。Chiong 等[27]应用比例边界多边形研究了功能梯度材料的断裂。Khudari 等[161]提出了一种模拟弯曲纤维复合材料的数值方法，可以在弯曲纤维附近自动化生成和细分网格。Lin 等[22]研究了分层复合材料三明治板。Hajian 等[162]研究了石墨烯的随机断裂分析。Liu 等[23]研究了功能梯度三明治板的自由振动和瞬态分析。Du 等[163]研究了功能梯度材料形成的梁的动力特性。Zhang 等[164]基于 SBFEM 多边形和多面体单元研究提出了一种模拟纤维复合材料的数值框架，通过在纤维与实体单元交接处插入节点的方式，实现了实体单元和杆单元的无缝耦合，并成功预测了纤维复合材料的力学性能。

2.11　并行计算

随着计算机并行计算性能的提高，数值分析对算法并行化的需求也越来越强。SBFEM 与八叉树网格结合，在并行计算领域具有独特的优势。Zhang 等[36]开发了一种基于 SBFEM 和八叉树网格的显式并行计算方法。该方法利用平衡八叉树网格的固定模板单元，减少内存需求和读写。得益于 Gravenkamp 等[165]提出的 SBFEM 质量阵对角化技术，可以避免求解线性方程组，从而避免了组装整体刚度阵，可在单元层面进行节点力计算，进而可以便捷、高效的并行化，充分利用预计算降低了单元计算成本。该算法被部署在澳大利亚国家计算中心的并行计算机集群 Gadi 上，调用超过 16 000 个 CPU 核实现了超过十亿自由度模型的大规模计算[36]（图 14）。Ankit 等[37]进一步完善该算法，研究了损伤问题的显式并行求解方法。Zhang 等[166]研究了异时步显式并行算法，可根据局部网格尺寸调整时间步。

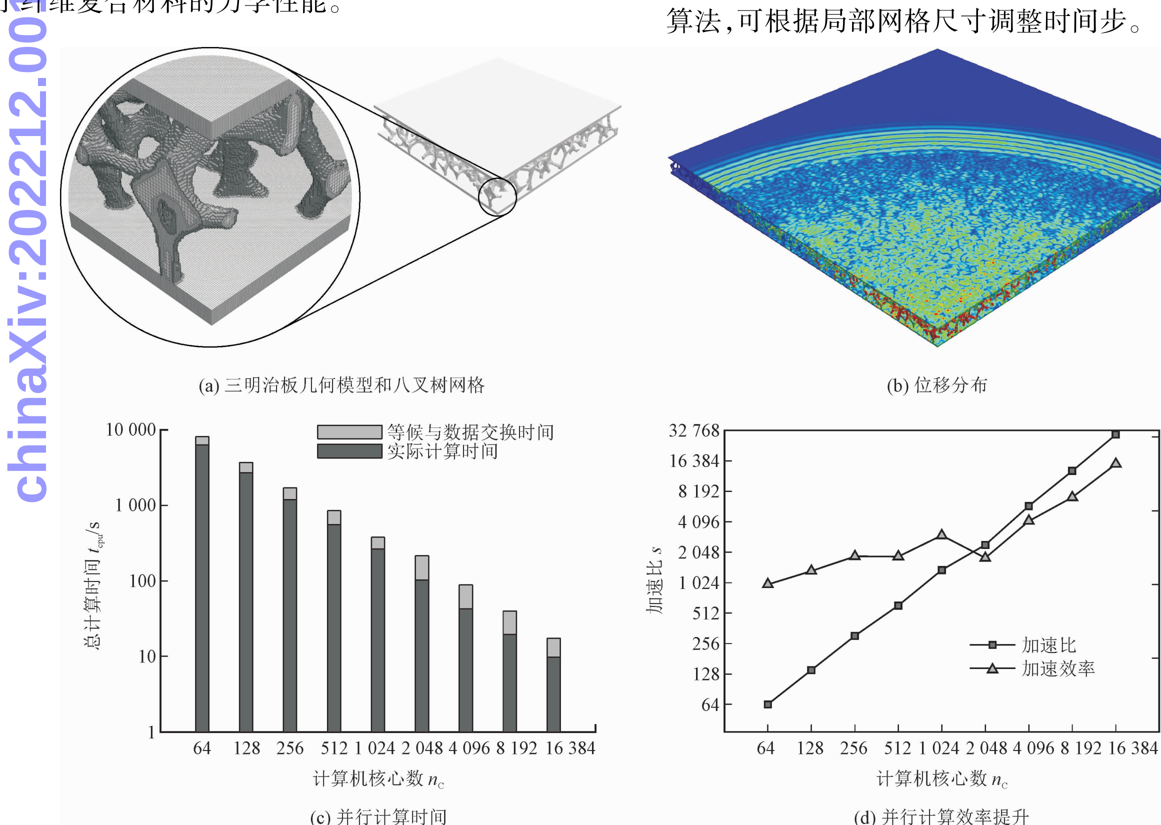


图 14　SBFEM 大规模并行计算[36]

Fig. 14　Massively parallel computing using SBFEM[36]

2.12　软件开发

近年来，将 SBFEM 单元与软件结合成为大量研究者的共识。Ya 等[167]将 SBFEM 多面体单元开发为商用软件 ABAQUS 的用户自定义单元 UEL，充分利用 SBFEM 在单元形状方面的灵活性和 ABAQUS 完善的求解系统，优势互补，实现了与商用软件的无缝对接。Ye 等在 ABAQUS 中开发了二维 SBFEM 多边形

UEL[168]。Yang 等[169]开发了三维多面体网格的 UEL/VUEL,并应用于静力和动力分析。Zhou 等[170]将 SBFEM 多面体单元用于三维核电厂房的动力问题求解。孔宪京等[171]自主研发了大型岩土工程分析软件 GEODYNA7.0,集成了 SBFEM 方法,深入开展了高土石坝工程的精细化数值分析。

3 总结与展望

SBFEM 在过去二十年间得到了长足发展,理论研究越来越完善,应用越来越深入,与商用软件的结合越来越紧密。其独特的单元构造方式,精确的形函数,广泛的适用性,均使其在各类数值方法中独具特色。SBFEM 在未来的重要研究问题还包括:①移动边界问题的自动化分析;② 大变形和大位移等几何非线性问题;③新型复合材料的非线性本构;④时间和空间的高阶算法;⑤融合 GPU 和 CPU 技术的并行计算算法;⑥力热声电等多场耦合问题。

随着激光扫描,3D 打印等技术在数字化时代的广泛应用,从构件到结构以至城市尺度的数字孪生概念逐渐成熟,机器学习和人工智能算法即将开启新一轮技术革命,与其相匹配的 SBFEM 全自动建模、分析技术和出色的大规模并行计算能力必将在未来发挥重要的作用。

4 配套源代码

https://github.com/ChongminSong/Platypus (2D,MATLAB)

https://github.com/ShukaiYa/SBFEM-UEL (3D,Fortran)

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